Wavelets in digital image processing

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Agenda

1. image spectrum
2. FFT
3. filtering
4. JPEG / JFIF

DIP scheme

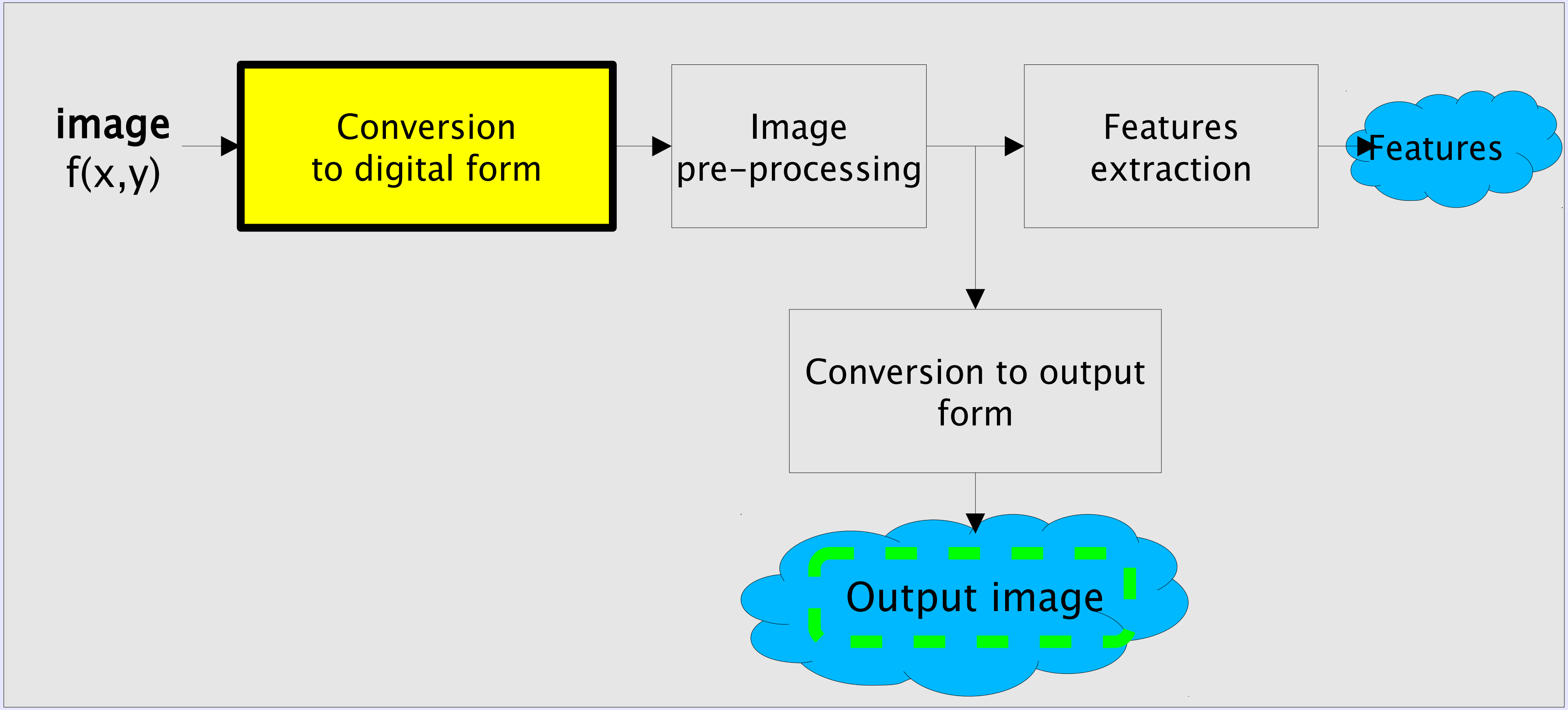


image
f(x,y)
Conversion
to digital form
Image
pre-processing
Features
extraction
Features
Conversion to output
form
Output image

Alfréd Haar

Alfréd Haar (Hungarian: Haar Alfréd; 11 October 1885, Budapest – 16 March 1933, Szeged) was a Jewish Hungarian mathematician. In 1904 he began to study at the University of Göttingen. His doctorate was supervised by David Hilbert.

The Haar measure, Haar wavelet, and Haar transform are named in his honor.

Between 1912 and 1919 he taught at Franz Joseph University in Kolozsvár. Together with Frigyes Riesz, he made the University of Szeged a centre of mathematics. He also founded the Acta Scientiarum Mathematicarum magazine together with Riesz.

His results are from the fields of mathematical analysis and topological groups, in particular he researched orthogonal systems of functions, singular integrals, analytic functions, partial differential equations, set theory, function approximation and calculus of variations.

Haar, A., Zur Theorie der orthogonalen Funktionensysteme, (Erste Mitteilung), Math. Ann. 69 (1910), 331–371 (at GDZ). (This is Haar's thesis, written under the supervision of David Hilbert.)

Before...

Untill 1930 in order to perform frequency analysis, scientists used Fourier Transform, where a function with a period equal to 2π can be decomposed into combination of **sin** and **cos** functions:

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

Coefficients a_0, a_k, b_k, could be expressed with:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx$$

Wavelets : history

1807	Fourier	Ortogonalna dekompozycja sygnału okresowego
1910	A. Haar	Alternatywna baza ortonormalna służąca do dekompozycji sygnału
1946	D. Gabor	Okienkowa transformata Fouriera
1965	Cooley Tuckey	Algorytm FFT (Fast Fourier Transform)
1976	C. Galand D. Esteban	Filtracja pasmowa (subband coding)
1976	Crochiere Zebber Flanagan	Kwadratowe filtry lustrzane (QMF)

1982	J.Morlet	Wprowadzenie pojęcia falek
1982	A. Grossman	Odwrotna transformata falkowa
1984	Morlet, Grossman	Badania matematyczne nad transformatą falkową i jej zastosowaniami
1985	Y. Meyer	Matematyczne podstawy teorii falek
1986	Daubechies Grossman Meyer	Konstrukcja rozwinięcia nieortogonalnego
1986	Meyer, Lemarie	Konstrukcja gładkiej (C^∞), ortonormalnej bazy falkowej na R i R^n
1986	Meyer, Mallat	Ogólny formalizm umożliwiający konstrukcję ortogonalnej bazy falkowej

1988	Daubechies	Konstrukcja rodziny gładkich falek ortonormalnych o zwartym nośniku przy zastosowaniu filtrów FIR
		Związki pomiędzy ciągłą transformatą falkową a jej dyskretnymi odpowiednikami na Z i Z^n
1989	Mallat	Algorytm dekompozycji i rekonstrukcji falkowej przy użyciu analizy wielorozdzielczej
1990	Alpert Rokhlin	Pierwsza konstrukcja multifalek
1992	Daubechies	Rodzina symetrycznych, ciągłych falek biortogonalnych o zwartym nośniku
1993	Newland	Harmoniczna transformata falkowa (HWT)

1994	Dahlke	Pierwsza konstrukcja falek sferycznych
1994	P. Schröder, W. Sweldens	Metoda konstrukcji falek sferycznych niezależna od parametryzacji
1996	G. Strang	Bardziej systematyczna teoria konstrukcji multifalek dających analizę wielorozdzielczą
1996	Foster	WWT (Weighted Wavelet Transform) Modyfikacja transformaty falkowej umożliwiająca analizę danych niekoniecznie równoodległych w czasie
2001	Daubechies, Guskov, Schröder, Sweldens	Falki jedno- i dwuwymiarowe określone na nieregularnych zbiorach punktów

Genesis

Waves: functions oscillating in time, in space or in both

Fourier Analysis is also a wave analysis (harmonic). It represents a signal using sin/cos waves.

It is suitable to perform frequency analysis of **stationary functions**.

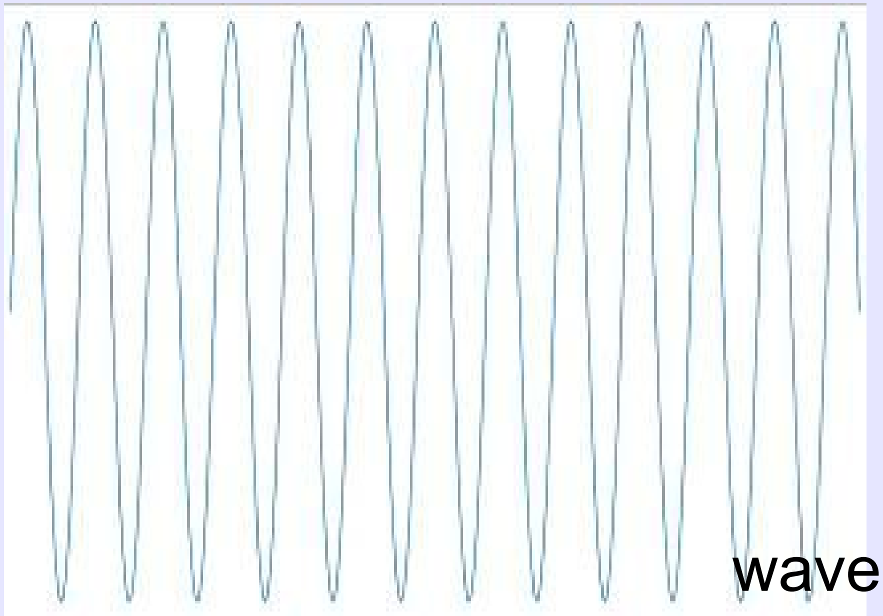
What is a wavelet?

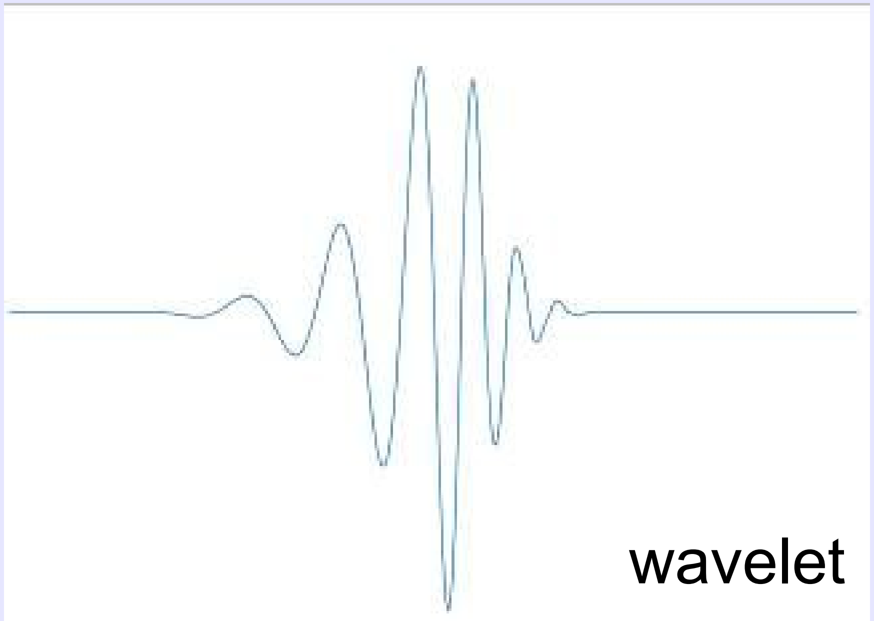
A wavelet is a **wave-like oscillation** with an amplitude that begins at zero, increases, and then decreases back to zero.

It can typically be visualized as a "brief oscillation" like one might see recorded by a seismograph or heart monitor.
Generally, wavelets are purposefully crafted to have specific properties that make them useful for signal processing.
Wavelets can be combined, using a "reverse, shift, multiply and integrate" technique called convolution, with portions of a known signal to extract information from the unknown signal.

Wavelet transform

Wavelet transform is similar to Fourier transform. Both utilize scalar product (convolution) of input signal *s(t)* and so called kernel (or mask).The difference is, hence, the kernel of the transformation.





Wavelet transform

Fourier transform uses **trigonometrical** functions (sin), as functions representing **single frequency**

On the other hand, wavelet transform uses **wavelets.** **There can be unlimited number of wavelets!**

Wavelet transform

Wavelet transformation is suitable for analyzing **non stationary signals**, because it provides a time-frequency information.

Often used interchangeably with the Fourier transform.

The reason for this is that as a result of wavelet transform we obtain information about the frequency of each signal spectral component with a finite precision.

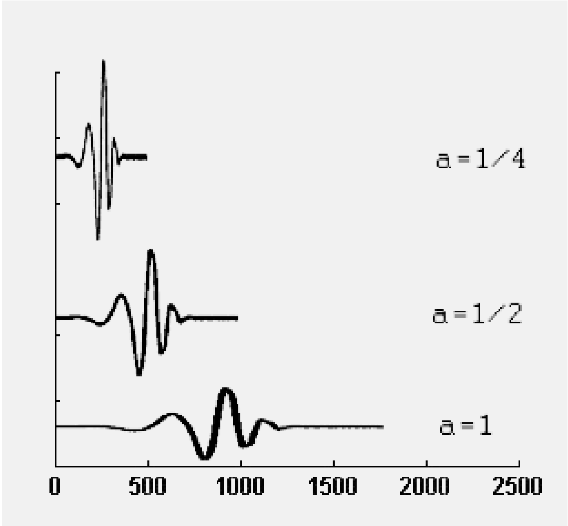
This is because the kernel transformation (ie function) does not represent an infinitely narrow frequency range but the frequency interval of width inversely proportional to the duration of wavelets.

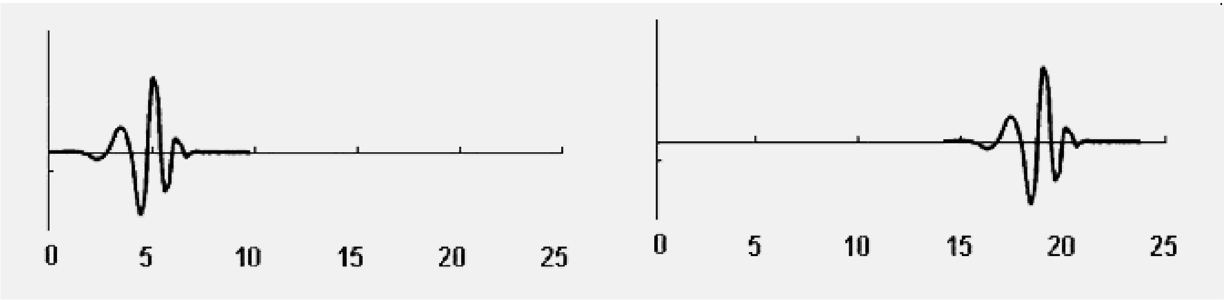
Definition

Wavelets are functions in the set of real numbers to the set of real numbers, each of which is derived from the mother using translation and scaling:

$$\Psi s,x(t) = \Psi(2s * t + x)$$

where: s,x – real numbers, Ψ – mother wavelet, Ψs,x – wavelet of scale s and translation x.





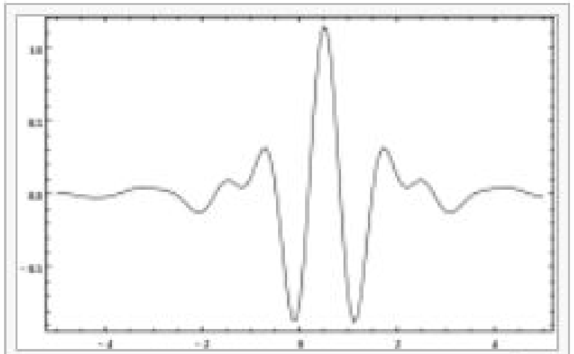
[wikipedia]

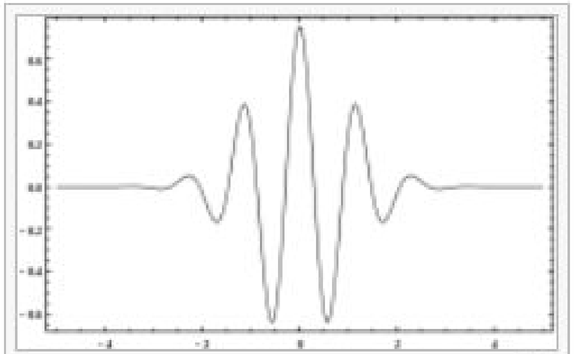
Definition (2/3)

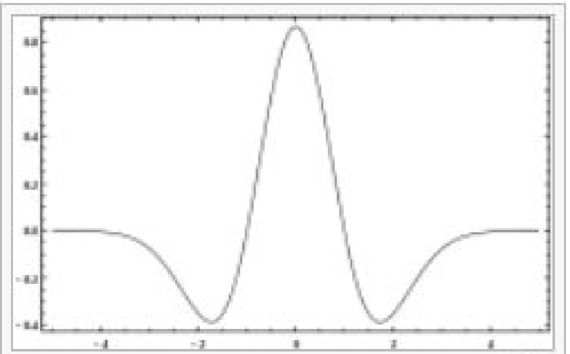
$$\psi^{s,x}(t) = \frac{1}{\sqrt{|s|}}\psi\left(\frac{t-x}{s}\right), \quad s \neq 0 .$$

$\int \psi(t)dt = 0$ At least several oscillations of mother wavelet (basis function)

$f = \sum_{m,n \in C} c_{m,n}(f)\psi_{m,n}$ Decomposition of *f* using base function


Meyer

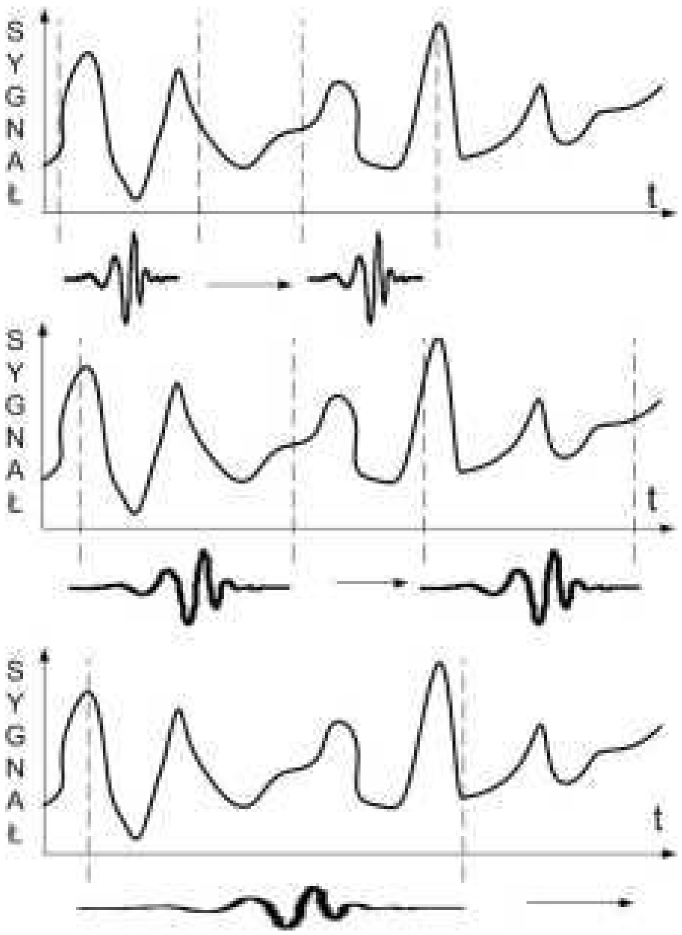

Morlet


Mexican Hat

[wikipedia]

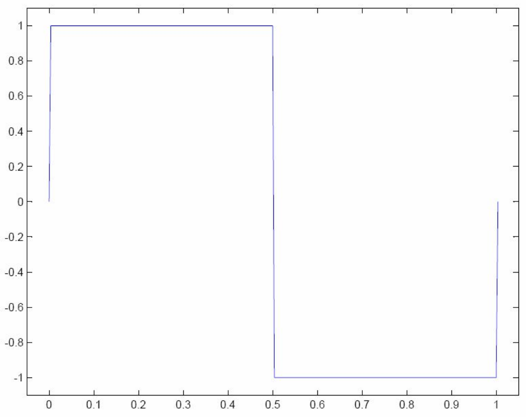
Definition (3/3)

The sum of the weighted functions Ψs,x can represent with any accuracy any continuous function like the cosine functions of different periods allow to represent any periodic function with arbitrary precision

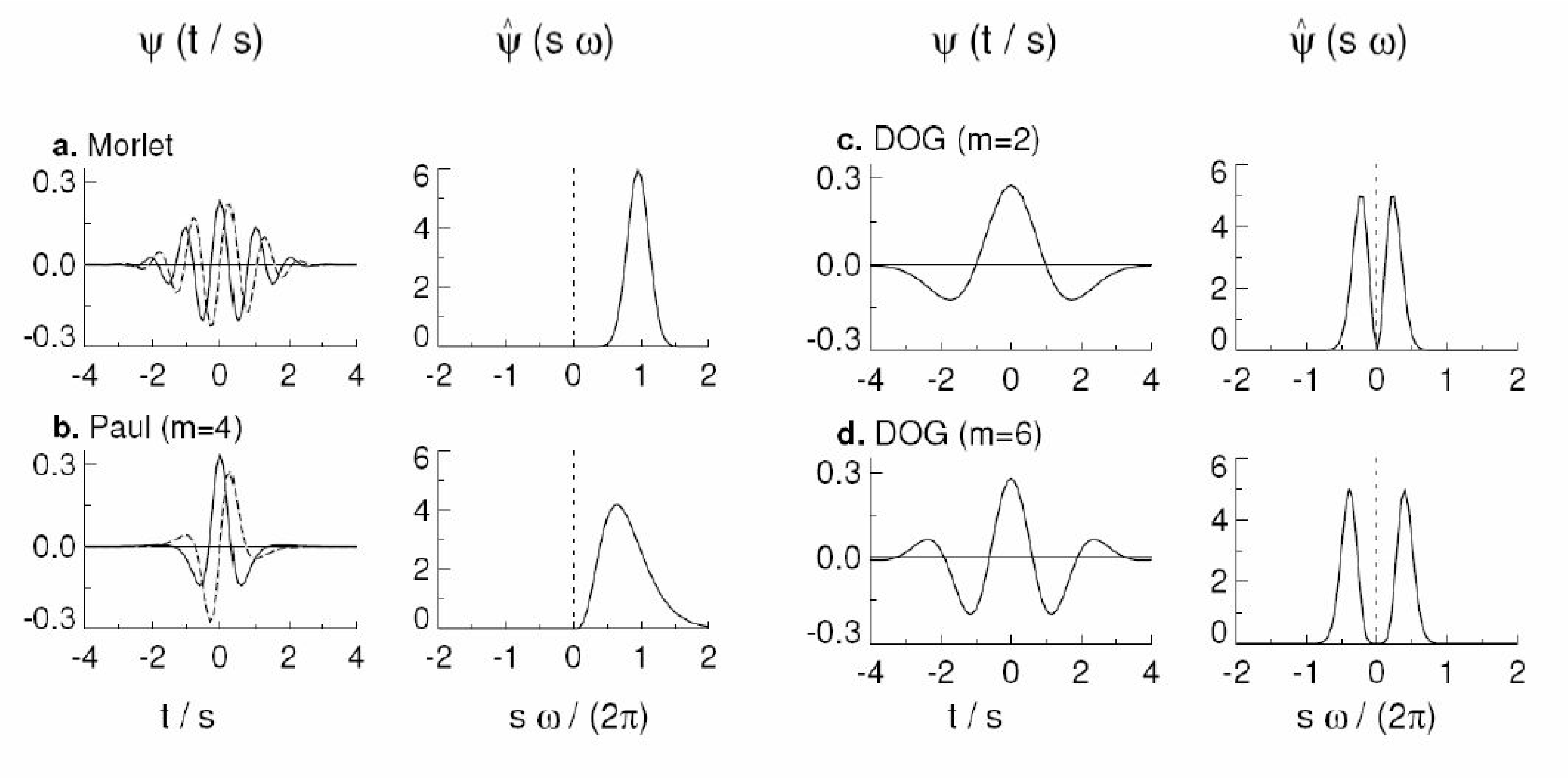


Assumptions

- Mean value = 0 $$\int_{-\infty}^{+\infty} \psi(t)\, dt = 0$$
- normalization $||\psi|| = 1$
- Set around $t=0$
- Finite range of transmision



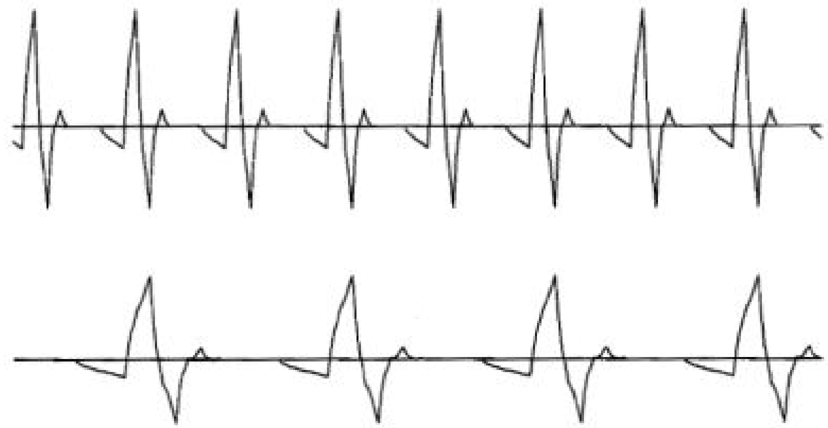
Examples


ψ (t / s)
ψ̂ (s ω)
ψ (t / s)
ψ̂ (s ω)
a. Morlet
b. Paul (m=4)
c. DOG (m=2)
d. DOG (m=6)
0.3
0.0
-0.3
6
4
2
0
-4 -2 0 2 4
-2 -1 0 1 2
t / s
s ω / (2π)

Wavelet family

- **translation** u and **scaling** s *of* mother wavelet

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$



- **normaliation** $\|\psi_{u,s}\| = 1$

Continuous (Integral) Wavelet Transform, CWT (IWT))

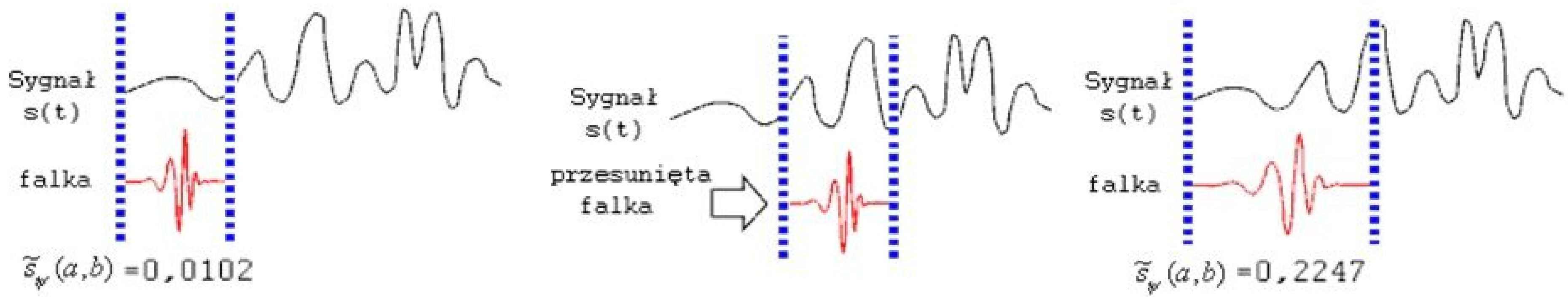
$$Wf(u,s) = \langle f, \psi_{u,s} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t-u}{s} \right) dt$$

Is a measure of difference of $f(t)$ in the neighborhood of u proportional to s

As a result we have a similarity measure of a signal and a wavelet

Example

As a result we have coefficients, dependent on *a* and *b* and a analyzed signal *s(t)*.



[wikipedia]

Inverse WT

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}$$

admissibility condition

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$$

No constant value

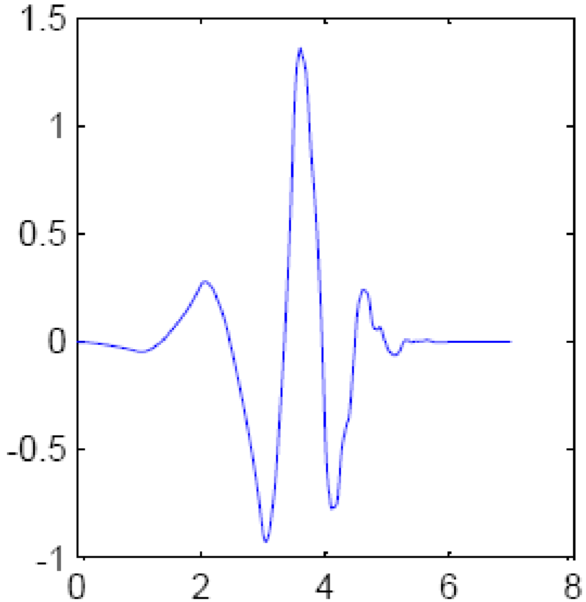
$$\hat{\psi}(0) = 0$$

Calderon, Grossmann, Morlet

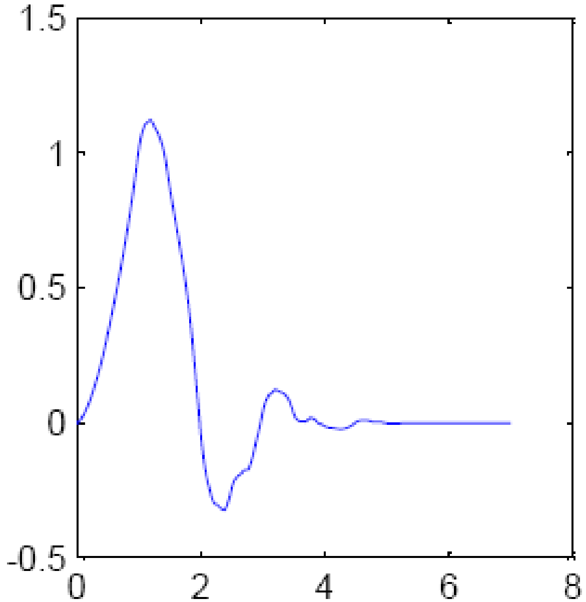
Scaling function

Mean value not zero!

$$||\phi|| = 1 \quad \lim_{\omega \to 0} |\hat{\phi}(\omega)|^2 = C_\psi$$

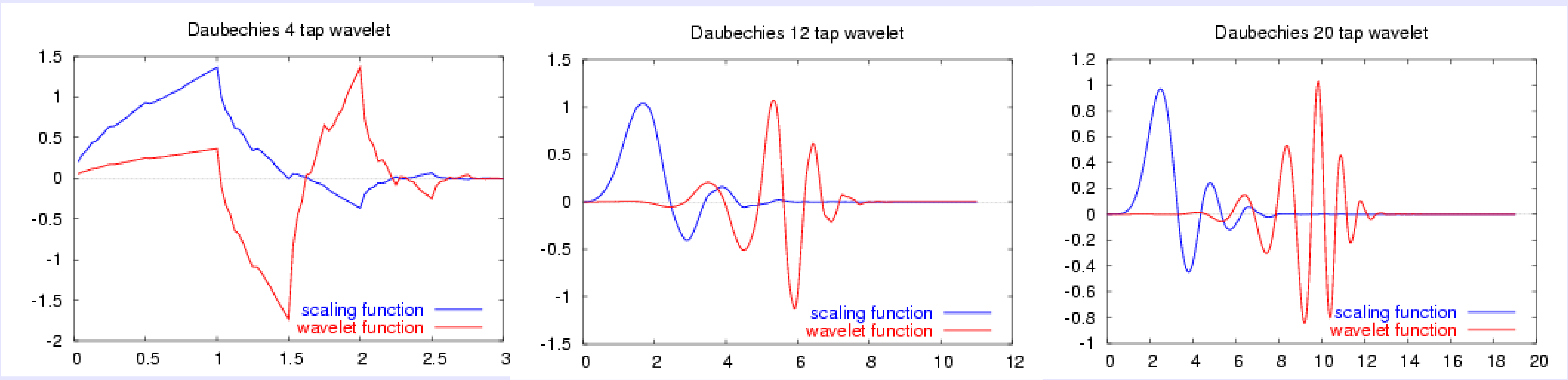


wavelet



Scaling function

Funkcja skalująca : przykłady



Wavelet analysis

A signal can be represented using a combination of wavelets and scaling functions.

$$f(t) = \sum_k c_j(k) 2^{j/2} \varphi(2^j t - k) + \sum_k d_j(k) 2^{j/2} \psi(2^j t - k)$$

$$c_j(k) = \langle f(t), \varphi_{j,k}(t) \rangle$$

$$d_j(k) = \langle f(t), \Psi_{j,k}(t) \rangle$$

$$s_{m+1}(t) = \sum_n c_{m+1,n} \varphi_{m+1,n}(t)$$

$$c_{m,n} = \sum_k h_{k-2n} c_{m+1,k}$$

$$d_{m,n} = \sum_k g_{k-2n} c_{m+1,k}$$

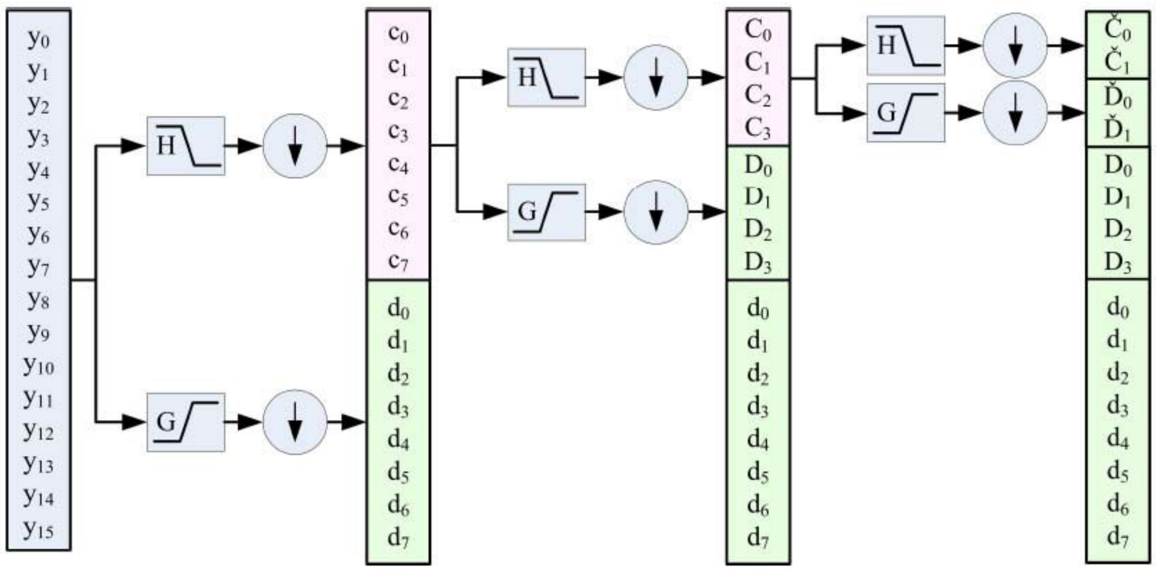
$$\mathrm{DWT} = \left\{ \{d_{m,n}\}_n, \{d_{m-1,n}\}_n, \ldots, \{d_{m-M,n}\}_n, \{c_{m-M,n}\}_n \right\}$$

Mallat's pyramid

Subsequent decomposition coefficients:

$$c_j(k) = \sum_m h(m-2k)c_{j+1}(m) \qquad d_j(k) = \sum_m g(m-2k)d_{j+1}(m)$$

h – low-pass filter H,
g – high-pass filter G.



Stéphane Mallat

•Input size = 2n, n ∈ N
•If not – then extend a signal

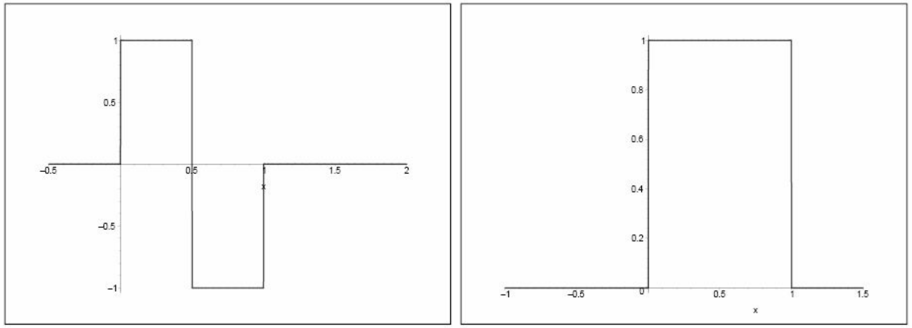
Haar Transform vs Fourier Transform

Fourier decomposition is not sufficient for many functions (i.e. “starcase function”), especially in the neighborhood of x=0, i.e.:

$$f(x) = \begin{cases} -\pi/2, & x \in (-\pi, 0), \\ 0, & x = 0, \\ \pi/2, & x \in (0, \pi) \end{cases}$$

Haar transform is better in such cases, since it is defined in the renge [0,1]:

$$H(x) = \begin{cases} -1, & 0 \leqslant x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leqslant x < 1 \end{cases}$$



Falka Haara ψ^H i funkcja skalująca φ^H

2D Haar Transform

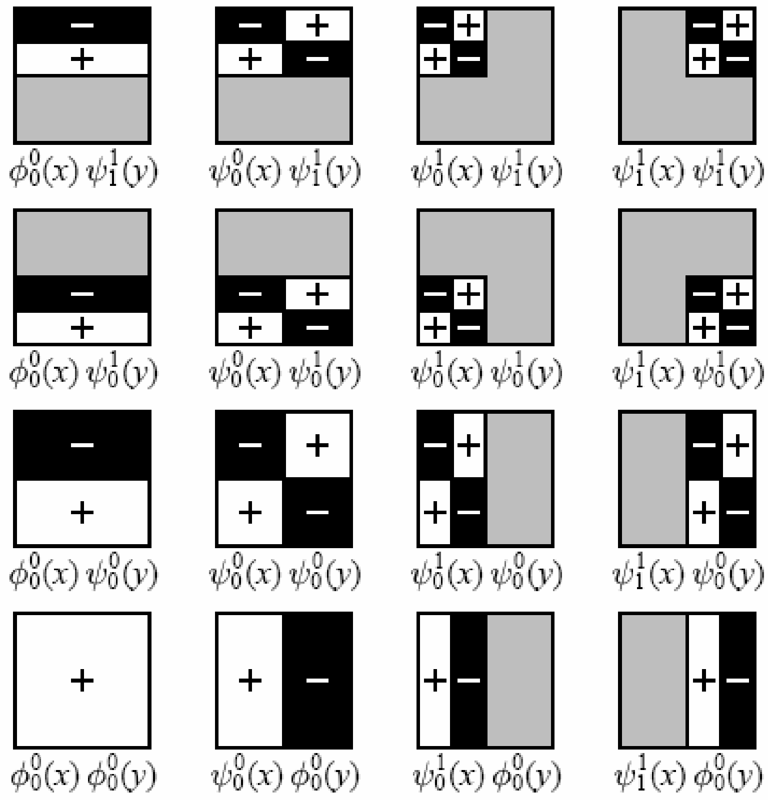


Figure 7 Standard construction of a two-dimensional Haar wavelet basis for V^2. In the unnormalized case, functions are +1 where plus signs appear, −1 where minus signs appear, and 0 in gray regions.

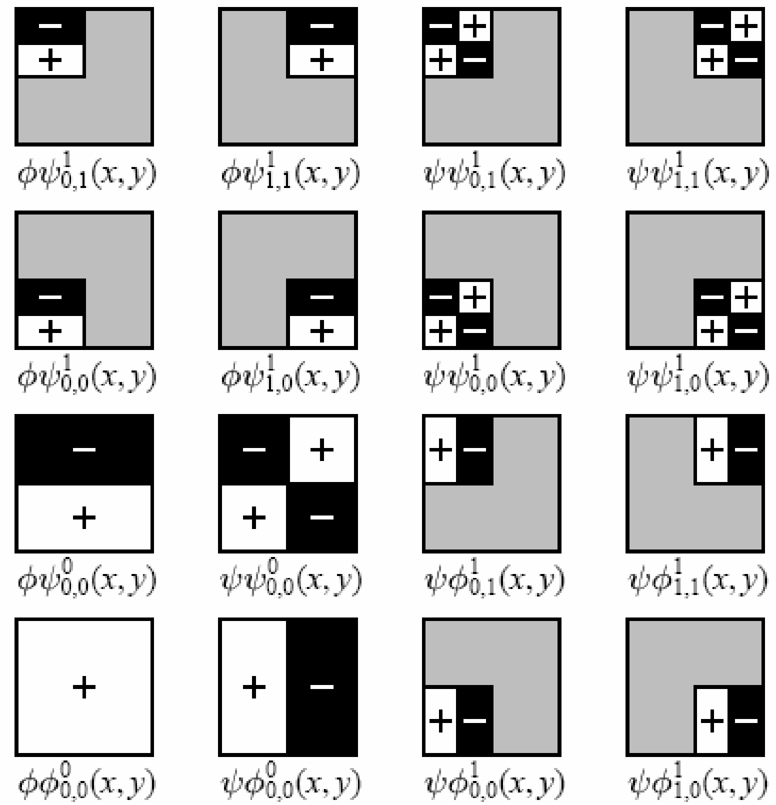
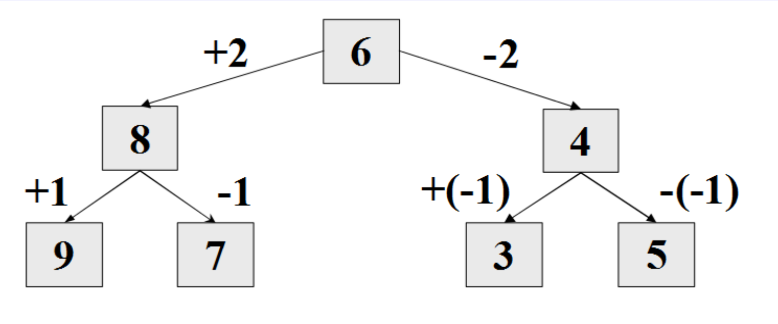


Figure 8 Nonstandard construction of a two-dimensional Haar wavelet basis for V^2.

1D Haar Transform : example

input: **9 7 3 5**
1. pair-wise averaging: **8 4**
2. transform coeff.: **1 -1**, since
   8+1 = **9** (1st el.)
   8-1 = **7** (2nd el.)
   4+(-1) = **3** (3rd el.)
   4-(-1) = **5** (4th el.)



Resolution	Average	Coefficients
4	9 7 3 5	
2	8 4	1 -1
1	6	2

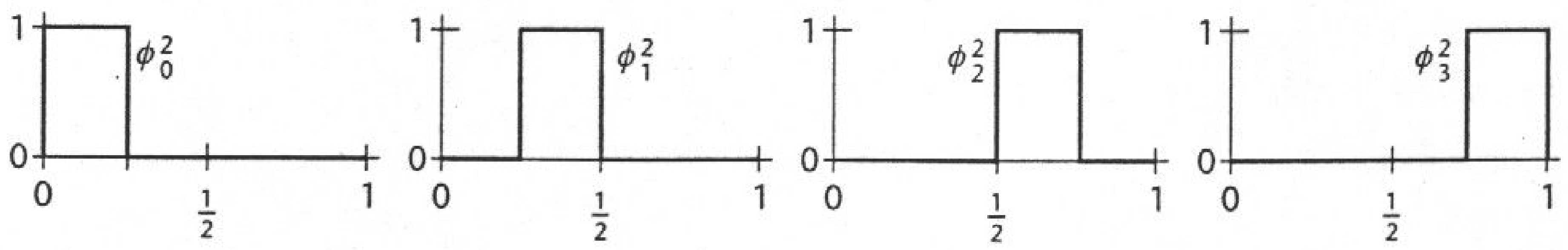
Output: **6 2 1 -1**

Haar Transform: box function

$$\phi_i^j(x) := \phi(2^j x - i), \quad i = 0, \dots, 2^j - 1$$

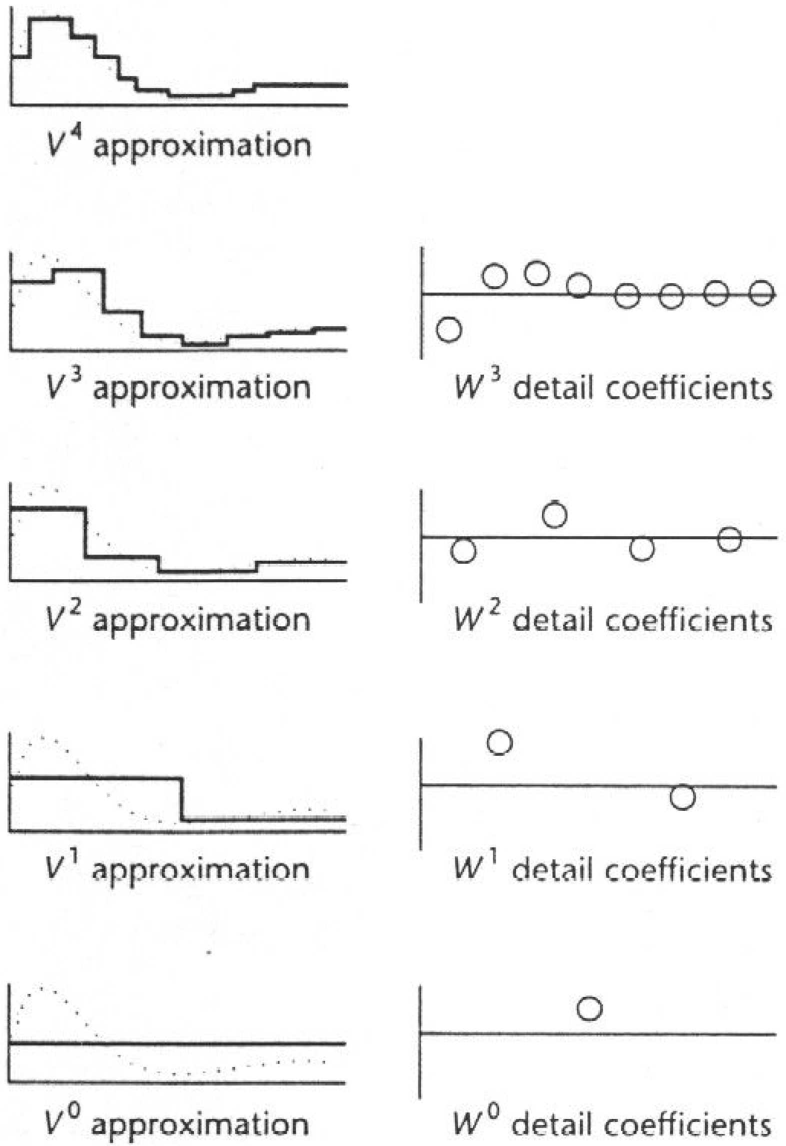
where

$$\phi(x) := \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$



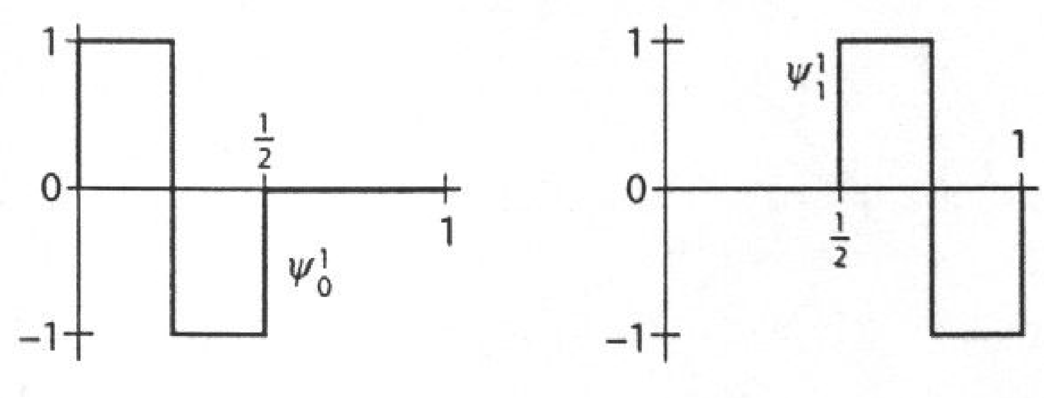
2 The box basis for V^2.

Haar Transform: box function



1 A sequence of decreasing-resolution approximations to a function (left), along with the detail coefficients required to recapture the finest approximation (right). Note that in regions where the true function is nearly flat, a piecewise-constant approximation works well, so the corresponding detail coefficients are relatively small.

Haar Transform



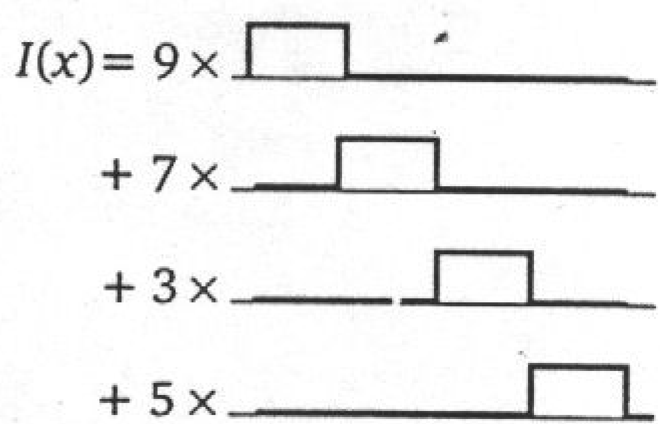
$$\psi_i^j(x) := \psi(2^j x - 1), \quad i = 0, \ldots, 2^j - 1$$

where

$$\psi(x) := \begin{cases} 1 & \text{for } 0 \le x < 1/2 \\ -1 & \text{for } 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

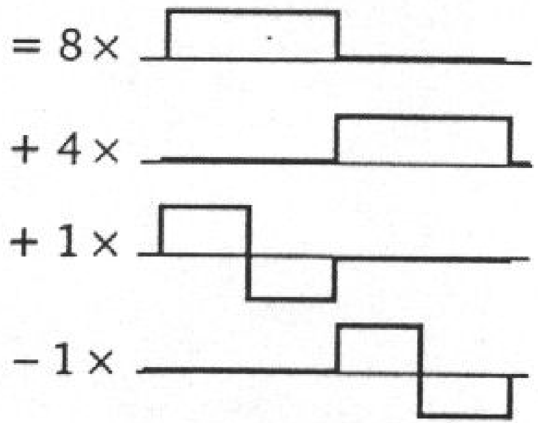
$$I(x) = c_0^2\phi_0^2(x) + c_1^2\phi_1^2(x) + c_2^2\phi_2^2(x) + c_3^2\phi_3^2(x).$$

A more graphical representation is

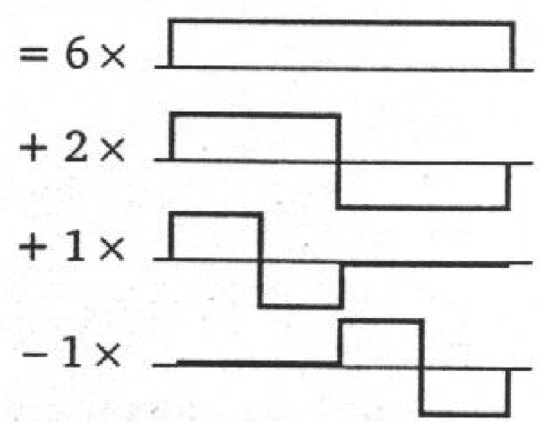


Haar Transform

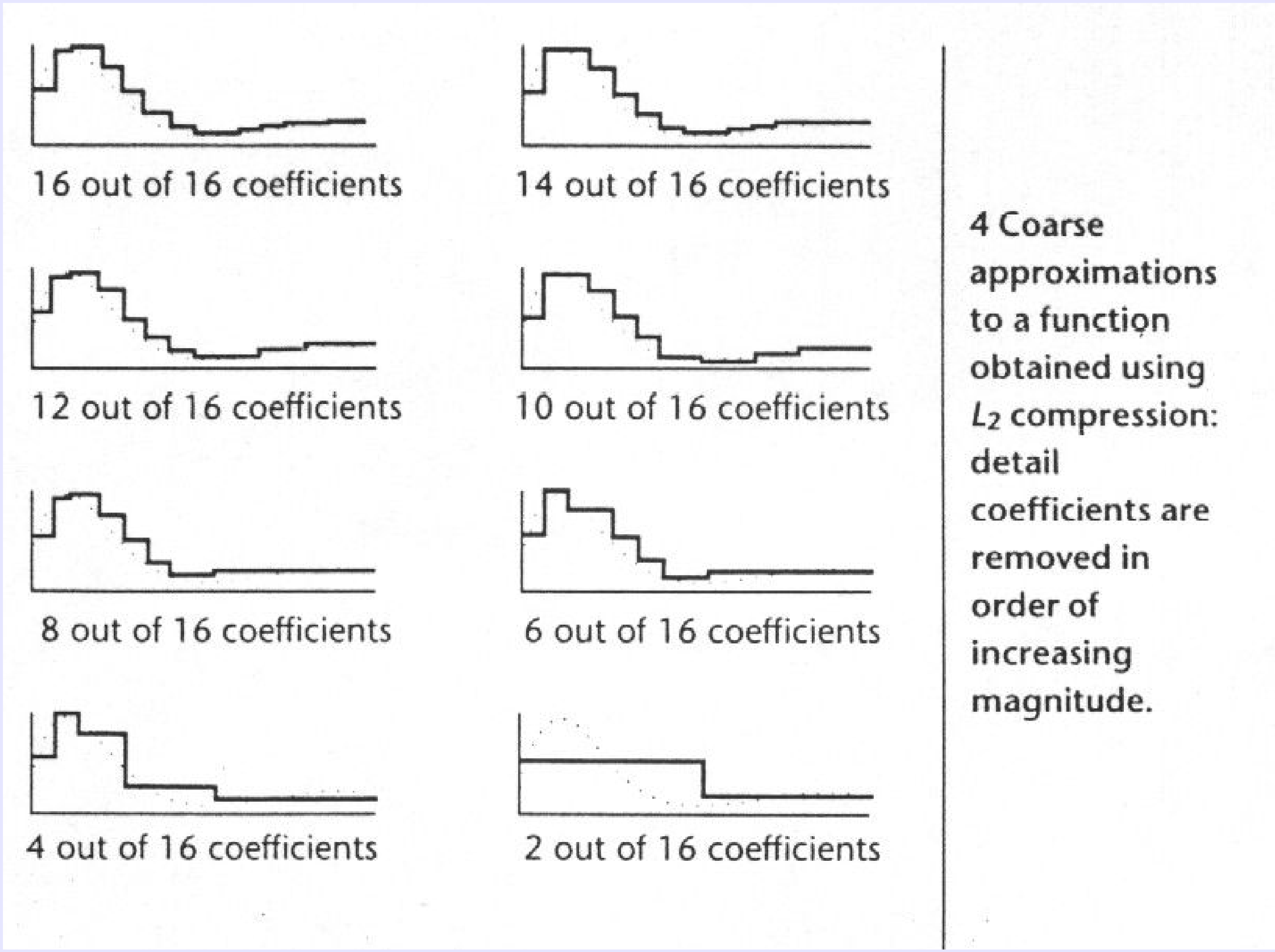
$$I(x) = c_0^1\phi_0^1(x) + c_1^1\phi_1^1(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x)$$



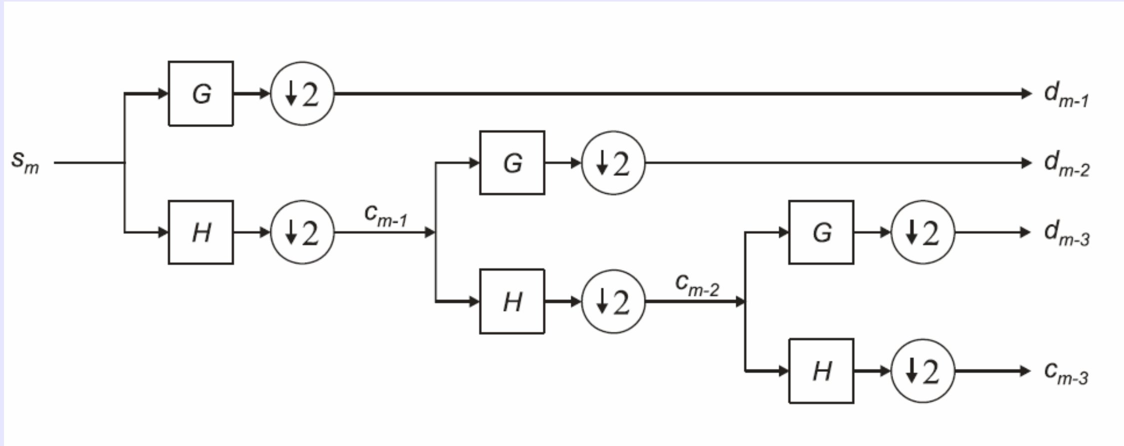
$$I(x) = c_0^0\phi_0^0(x) + d_0^0\psi_0^0(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x)$$



Haar Transform: compression



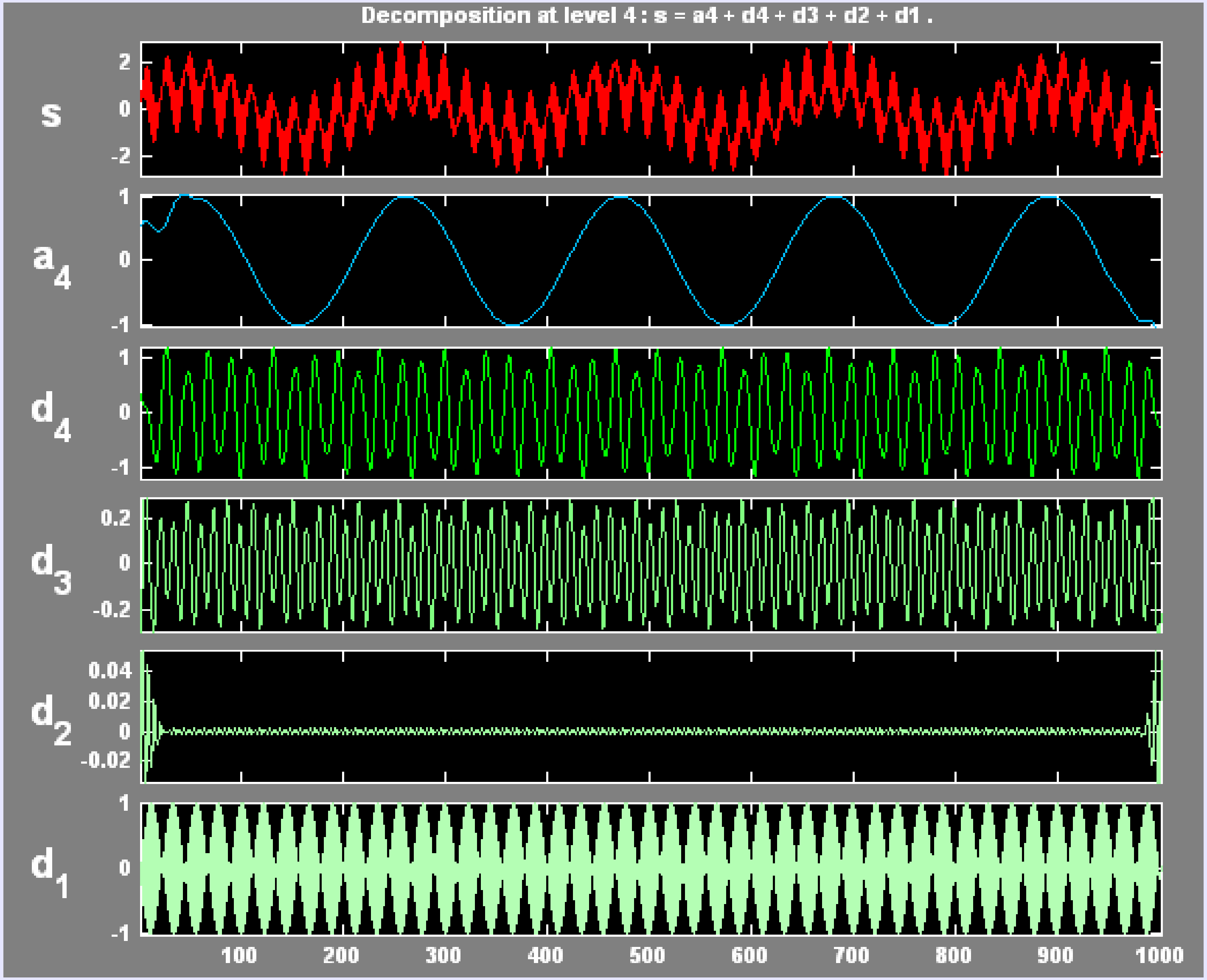
Wavelet decomposition



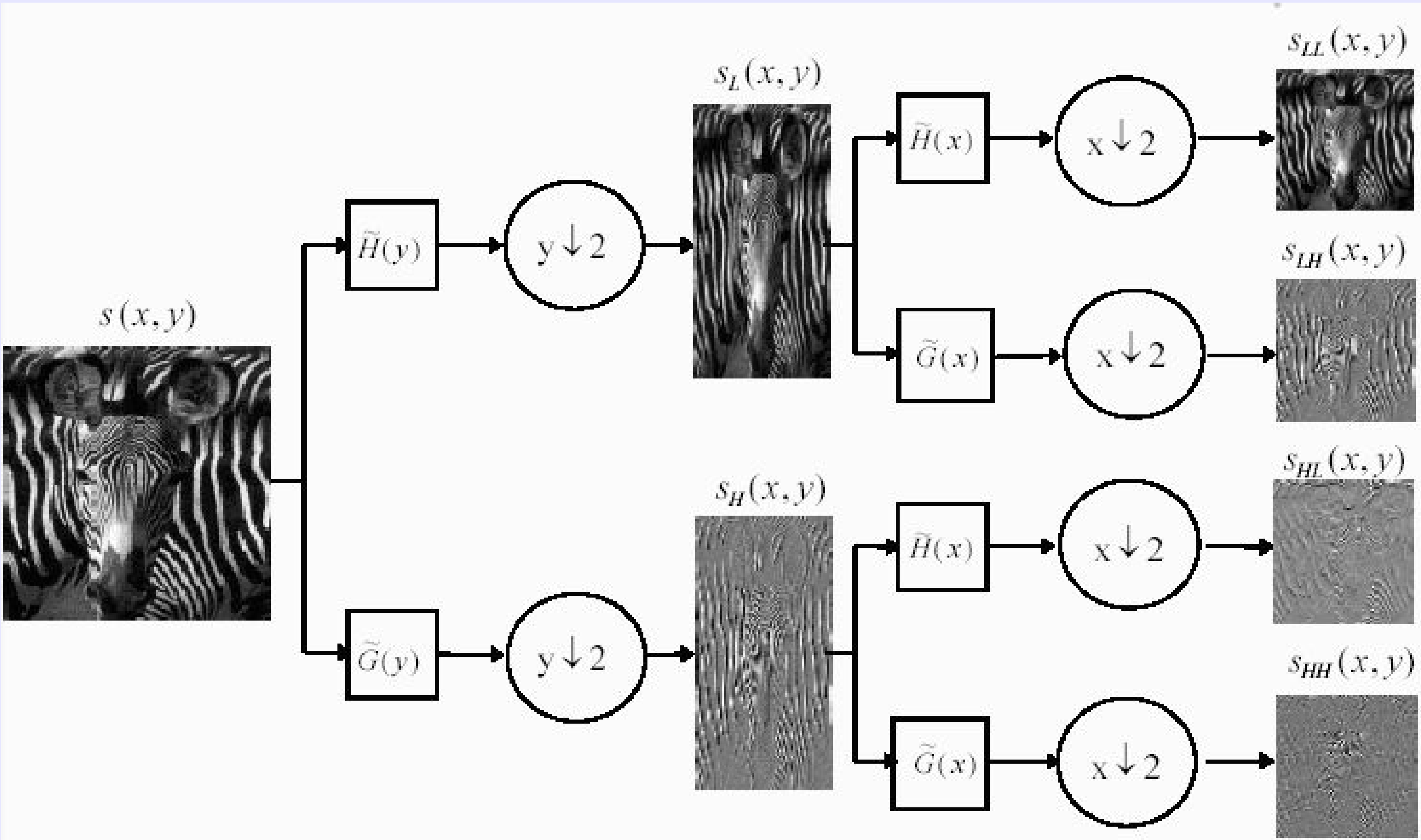
procedure *DecompositionStep*(C: **array** $[1\,.\,.\,h]$ **of reals**)
  for $i \leftarrow 1$ **to** $h/2$ **do**
    $C'[i] \leftarrow (C[2i-1] + C[2i])/\sqrt{2}$
    $C'[h/2+i] \leftarrow (C[2i-1] - C[2i])/\sqrt{2}$
  end for
  $C \leftarrow C'$
end procedure

procedure *Decomposition*(C: **array** $[1\,.\,.\,h]$ **of reals**)
  $C \leftarrow C/\sqrt{h}$ *(normalize input coefficients)*
  while $h > 1$ **do**
    DecompositionStep($C[1\,.\,.\,h]$)
    $h \leftarrow h/2$
  end while
end procedure

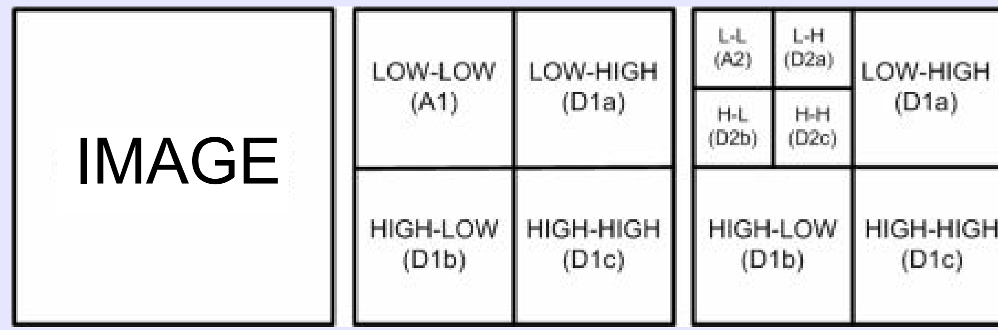
Wavelet decomposition



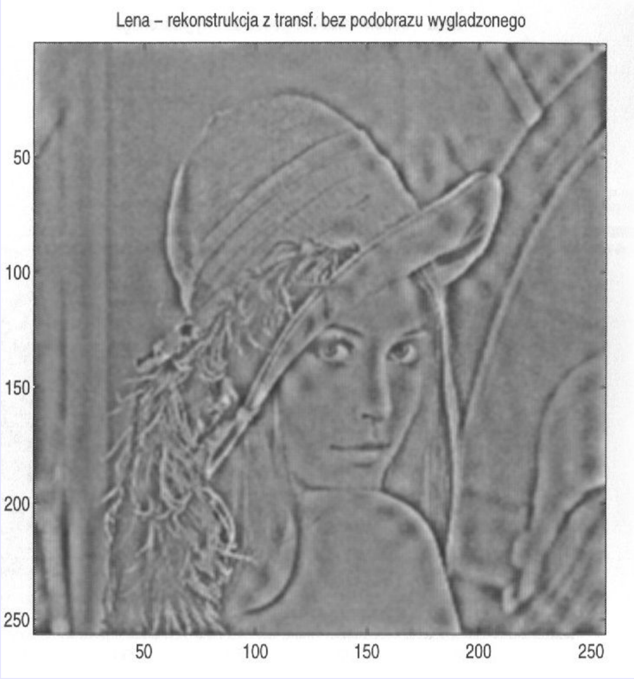
Wavelet image decomposition

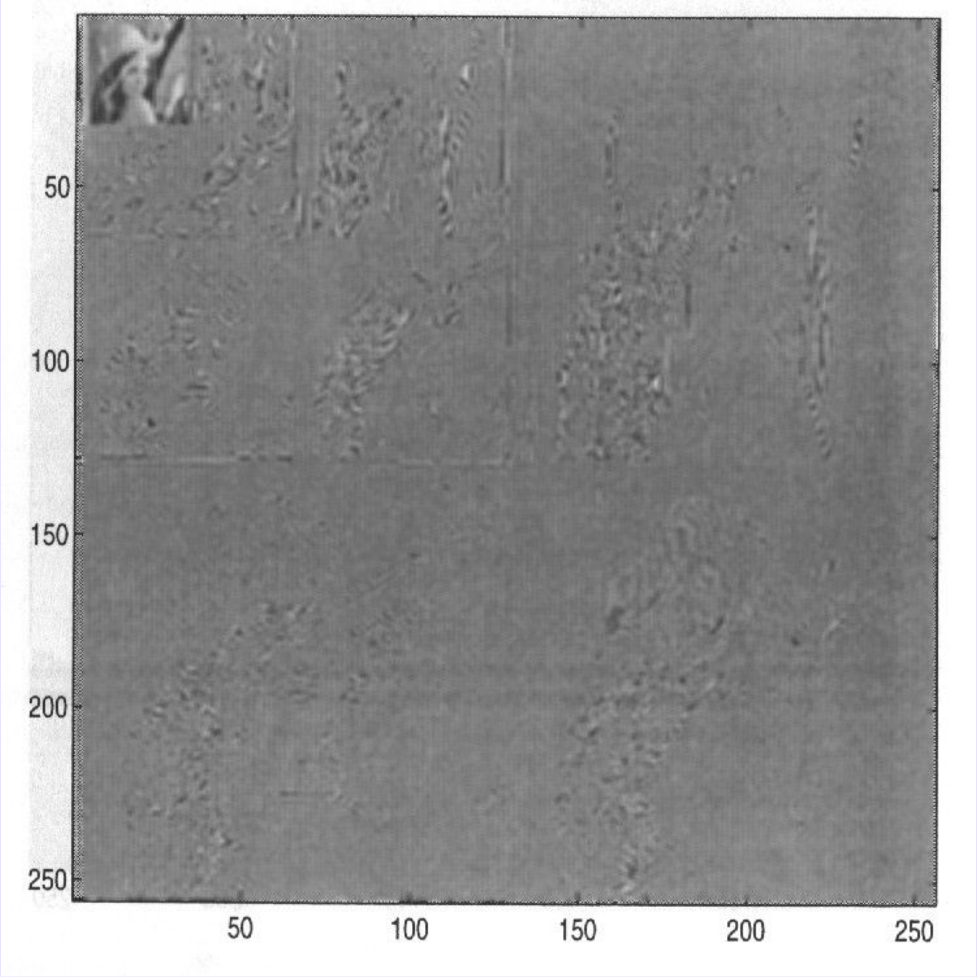


Wavelet image decomposition



Wavelet image decomposition





Applications

- Gravity waves detection (CWT),
- Sun activity evaluation (CWT),
- JPEG2000,
- digital watermarking,
- monitoring of ships movements,
- paitings style expression (van Gogh, Picasso, Monet, Klee i in.),
- seismic data exploration (CWT i DWT),
- integral/differential equations,
- radar data denosing and recognition (SAR, Synthetic Aperture Radar),

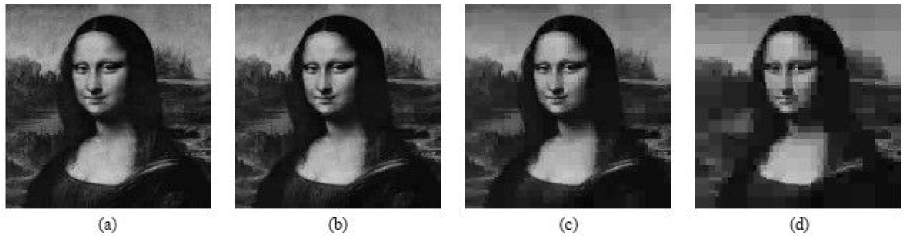
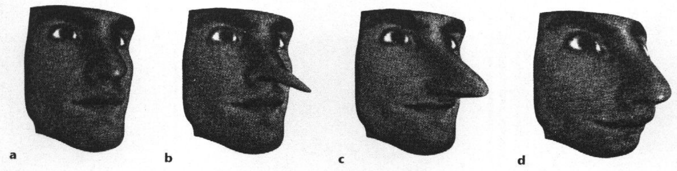
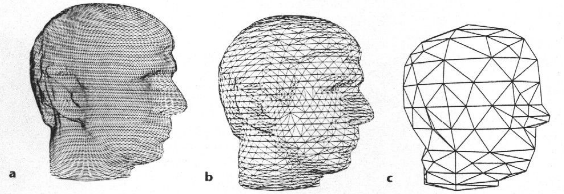


Figure 9 L^2 wavelet image compression: The original image (a) can be represented using (b) 19% of its wavelet coefficients, with 5% relative L^2 error; (c) 3% of its coefficients, with 10% relative L^2 error; and (d) 1% of its coefficients, with 15% relative L^2 error.

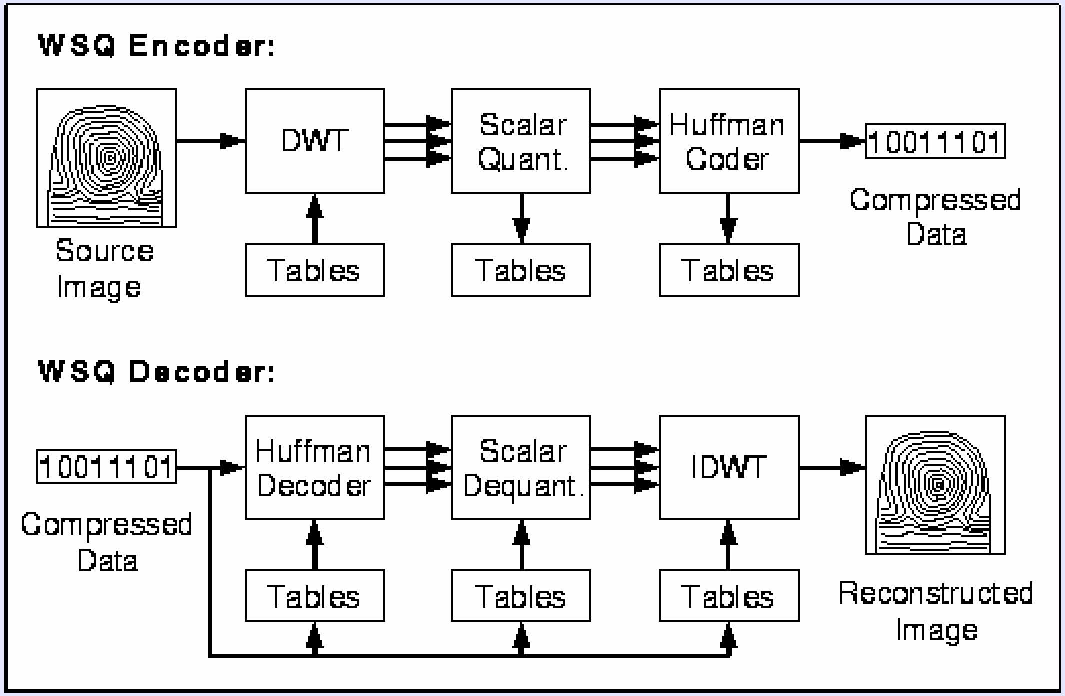


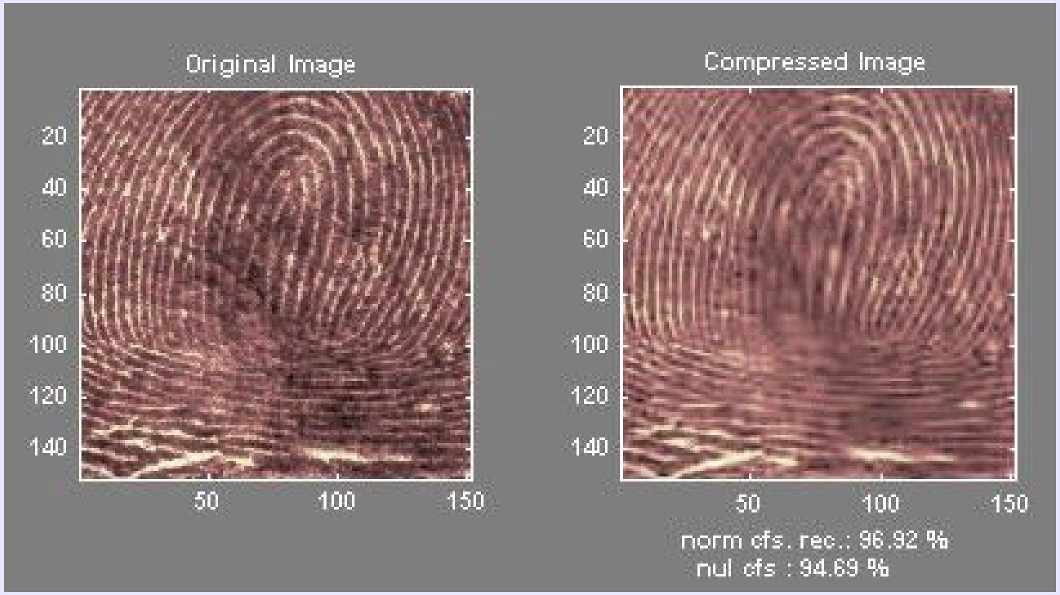
6 Surface manipulation edited at different levels of detail: (a) the original surface, at (b) a narrow scale, (c) an intermediate scale, and (d) a broad scale.



7 Surface approximation using sub-division surface wavelets: (a) the original surface, (b) an intermediate approximation, and (c) a coarse approximation.

Compression



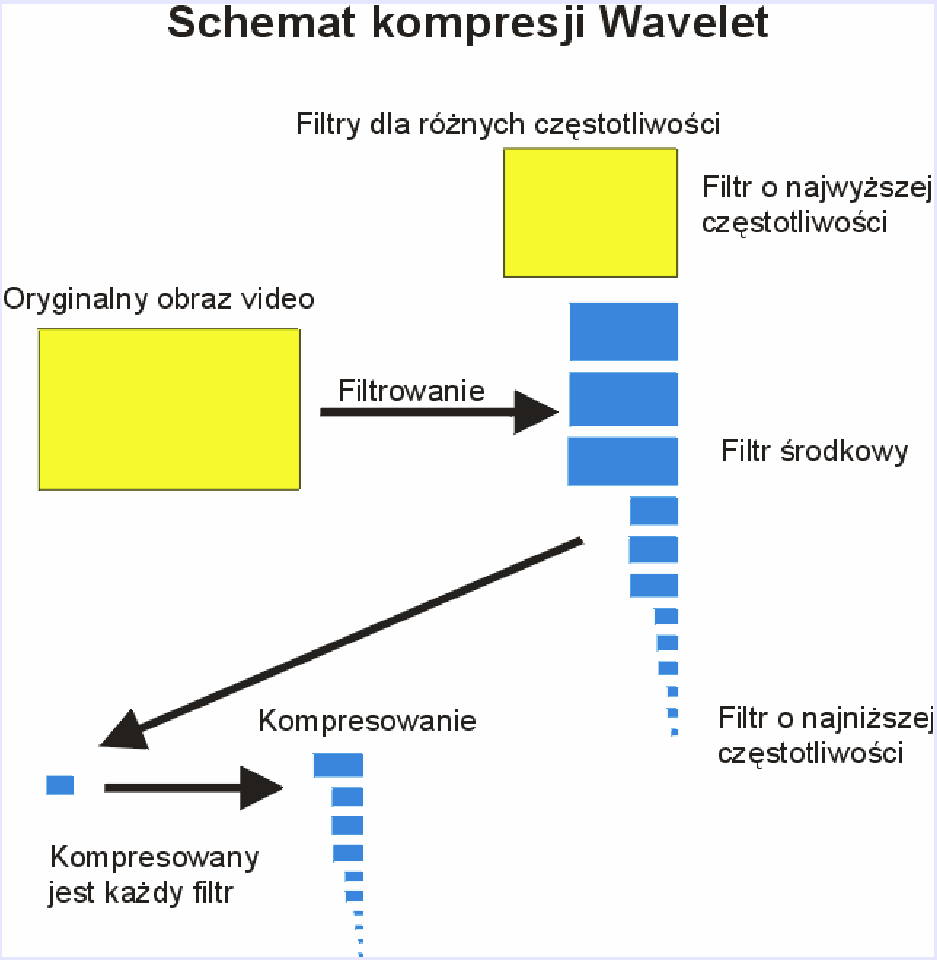


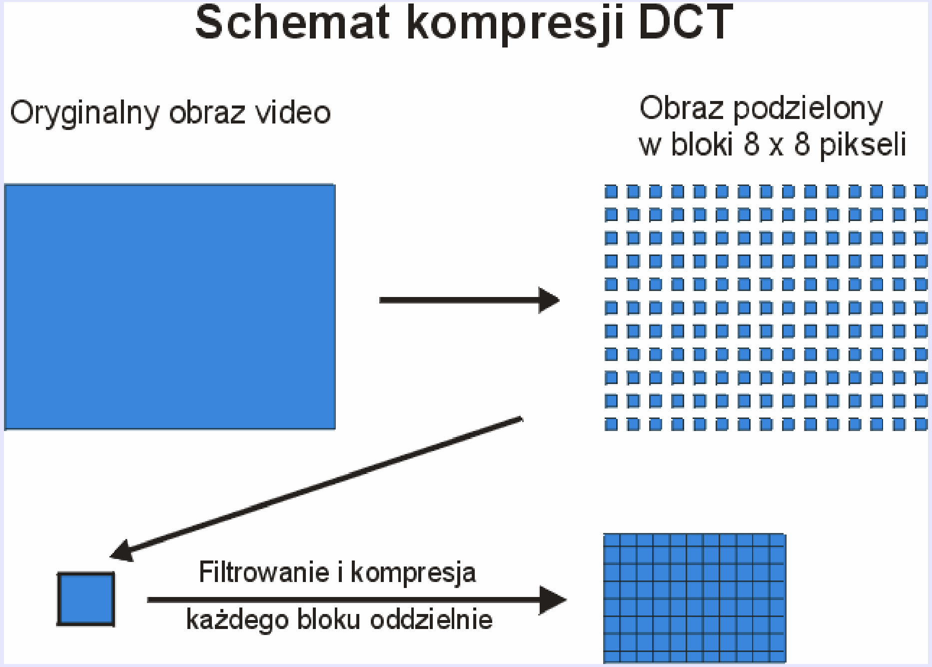
Before DWT, an image is decomposed into 64 bands using 2D DWT (iteratively)

After DWT, data (float/double) are quantized/cutoff

Then, data are coded using Huffmann coding or other entropy-based coder

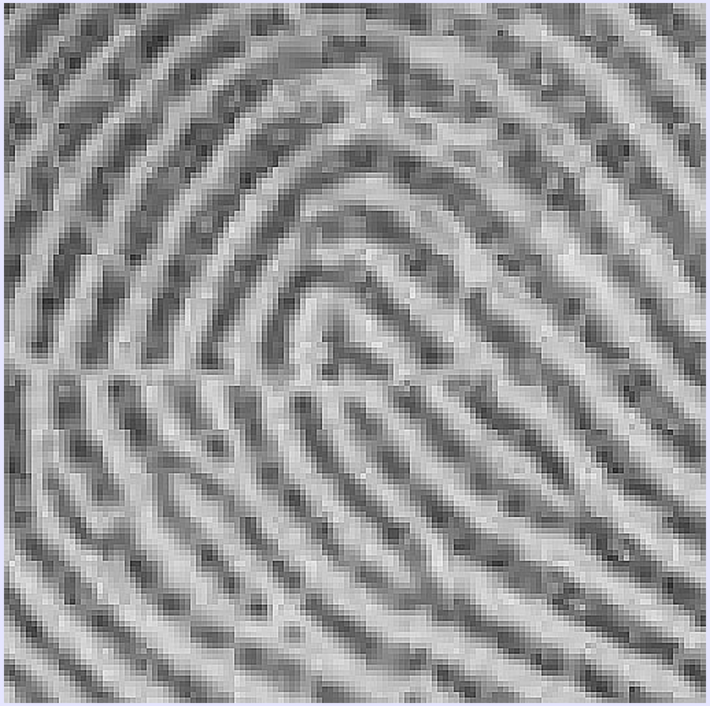
Wavelets/DCT compression





Compression

JPEG image;
file size 45853 bytes,
compression ratio 12.9.



WSQ image;
file size 45621 bytes,
compression ratio 12.9.